Natural Resonant Frequencies of Microwave Dielectric Resonators

The application of dielectric resonators are currently of considerable interest in microwave techniques [1], [2]. The design of a dielectric resonator, as in the case of a metal cavity, depends on its natural resonant frequencies. Since exact solutions of dielectric resonators having shapes other than a sphere [3] or a doughnut [4] cannot be rigorously computed, approximate techniques must be adopted to solve the problem. Two types of resonant modes can be excited in a dielectric resonator, namely, the H mode and the E mode. The H mode is defined as the mode which has a large normal component of magnetic field at the boundary surfaces; and the E mode is the mode with no predominant normal component of magnetic field at the surfaces.

The electromagnetic fields in dielectric resonators of high permittivity approximately satisfy the open-circuit boundary condition. This can be verified by considering a plane wave propagating in the dielectric, incident on the boundary between dielectric and air [5]. The open-circuit boundary (OCB) condition is defined as

$$n \times H = 0$$
$$n \cdot E = 0$$

where n is the unit vector normal to the boundary. In other words, at the OCB, the normal component of electric field and the tangential component of magnetic field vanish. In fact, all E modes and some higher order H modes satisfy the OCB condition very well if the relative permittivity is much larger than unity. This can be seen by comparing a spherical dielectric resonator and a perfect OCB spherical resonator of the same size and the same permittivity. For a relative permittivity $\epsilon_r = 100$, the differences of resonant frequencies of the E modes is smaller than 1 per cent, the H_{2mn} (where the subscripts indicate the number of variations in the spherical coordinate r, θ, ϕ direction, respectively) modes differ approximately by 3 per cent. However, the difference is about 14 per cent for the H_{1mn} modes. Therefore, the resonant frequencies and the field distributions of dielectric resonators can be calculated, as an approximation, by assuming OCB condition; except some modifications are necessary for some lower order H modes.

Consider a homogeneous, lossless, circular cylindrical dielectric resonator of radius a and length L. The resonant frequencies are, approximately, given by the roots of the following equations [5]:

$$J_m'(\beta a) = 0, \quad \text{for } TE_{lmn} \text{ mode} \qquad (1)$$

$$J_m(\beta a) = 0, \quad \text{for } TM_{lmn} \text{ mode} \qquad (2)$$

where

$$\beta^2 = (2\pi/\lambda_0)^2[\epsilon' - (n\lambda_0/2L)^2] \qquad (3)$$

and λ_0 is the free space wavelength, J_m is the mth order Bessel's function of the first kind.

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TABLE I

THEORETICAL AND EXPERIMENTAL RESONANT FREQUENCIES OF CYLINDRICAL $SrTiO_3$ RESONATORS

a (mils)	L (mils)	Mode*	f_0(GC) OCB approximation	(Theoretical) modified OCB approximation	f(GC) measured
32.8	68	$TE_{10\delta}$	8 26	9 18	9.28
		$TE_{11\delta}$	13 15	13.9	13.827
		TM_{101}	14 0	/	13.66
32.25	55.3	$TE_{10\delta}$	8.39	9.45	9.71
		$TE_{11\delta}$	13.36	14.15	14.24
		TM_{101}	14.5	/	14.203
35.2	30.6	$TE_{10\delta}$	7 68	10 27	10 43
		$TE_{11\delta}$	12 24	14 46	14 335
		TM_{101}	16.10	/	16.23

* $TE_{10\delta}$ and $TE_{11\delta}$ are H modes, TM_{101} is E mode.

The subscripts, l, m, n, denote the number of variations of the fields in the ρ, ϕ, z-direction respectively. Note that the TE_{lm0} and the TM_{1mn} modes need some modifications.

To obtain a better solution for the resonant frequencies of the TE_{lm0} modes, it is assumed that only the circular surface satisfies the OCB condition. The ϕ-component of electric field inside the resonator is then

$$E_\phi{}^i = A_i(z)J_m(\beta\rho)\cos m\phi \qquad (4)$$

The field immediately outside the resonator has the same distribution as that of the field just inside the resonator. Hence, the electric field just outside the resonator may be expressed as

$$E_\phi{}^0 = A_0(z)J_m(\beta\rho)\cos m\phi \qquad (5)$$

Substituting (4) and (5) into Maxwell's equations and matching the solutions at $z = \pm L/2$ yield

$$\zeta_0 = \zeta \tan(\zeta L/2) \qquad (6)$$

$$\zeta^2 + \epsilon_r\zeta_0{}^2 = (\epsilon_r - 1)\beta^2 \qquad (7)$$

where

$$\zeta^2 = (2\pi/\lambda_0)^2\epsilon_r - \beta^2 \qquad (8)$$

$$\zeta_0{}^2 = \beta^2 - (2\pi/\lambda_0)^2 \qquad (9)$$

and β is given by (1). The two simultaneous equations (6) and (7) can be solved for ζ and ζ_0 graphically or by a computer. With the knowledge of ζ or ζ_0, the resonant frequency can be computed by (8) or (9). Physically, the resonant mode is no longer the TE_{lm0} mode, since the field has a fraction δ of one-half cycle sinusoidal variation along the z direction [1], where $\delta = L\zeta/\pi$. A similar modification for TM_{1mn} mode can be made by assuming that only two flat surfaces satisfy the OCB condition. The resonant frequencies are approximately given by [5]

$$\frac{J_m'(\beta a)}{J_m(\beta a)} + \frac{\beta}{\epsilon_r\alpha}\frac{K_m'(\alpha a)}{K_m(\alpha a)} = \pi^2(kL\beta a)^{-2}\frac{J_m'(\beta a)}{J_m''(\beta a)} + \beta^3\alpha^{-3}\frac{K_m'(\alpha a)}{K_m''(\alpha a)} \qquad (10)$$

where

$$\alpha^2 = (n\pi/L)^2 - (\pi/\lambda_0)^2$$
$$\beta^2 = (2\pi/\lambda_0)^2\epsilon_r - (n\pi/L)^2$$
$$k^2 = (2\pi/\lambda_0)^2\epsilon_r$$

K_m is the modified Bessel's function of the second kind. Equation (10) can be solved for λ_0 by computer. The method outlined above is applicable to rectangular and elliptic cylindrical dielectric resonators.

Measured results made on cylindrical $SrTiO_3$ ($\epsilon_r = 279$) resonators agree well with the theoretical values. They are listed in Table I.

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A Note on Varactor Frequency Tripler

Several papers [1]–[9] in the past ten years have considered analysis of harmonic generators with a nonlinear reactance. The most comprehensive work on the varactor multipliers is by Penfield and Rafuse [10], in which a detailed analysis of the frequency tripler has been made using abrupt-junction varactors. Leonard [9] has made a detailed analysis of varactor frequency doublers to predict the power and efficiency for nonlinearities ranging from graded-junction to hyperabrupt-junction varactors.

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